ANISOTROPIC THEORY OF GROWTH STRESSES IN TREES

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SUMMARY

Growth stresses are a consequence of the growth process in trees; they arise from stresses which develop in each new peripheral sheath of wood cells. This situation is analysed using a linearly elastic, three-dimensional orthotropic model for wood. Expressions for the stress and strain distributions in a tree are derived. Some previous experimental work is reinterpreted by analysing the stress and strain redistributions that occur during various cutting processes. It is shown that the discrepancy in the qualitative behaviour of the radial strain between previous theories and experimental measurements may be accounted for by the redistribution of stresses that occurs during the preparation of a cross-cut disc in the experiments.

INTRODUCTION

Earlier investigators (JACOBS 1938, 1939, 1945 and BOYD 1950a) have shown the presence of large internal stresses in trees which can be as much as 10^7 N.m.$^{-2}$ (about 1500 p.s.i.) or more. It was established that these stresses were generated by the growth processes occurring in a tree and they therefore became known as "growth stresses". Although these stresses can not normally be observed directly in a standing tree they are manifested by the warping that occurs when the stresses are released during timber cutting. This can lead to pinching of the saw and also to heart shakes in a freshly felled tree (BOYD 1950b). JACOBS (1965) presents some excellent photographs of the warping that occurs when both gymnosperms (softwoods) and angiosperms (hardwoods or broad-leaved species) are sawn up. The phenomenon is particularly marked in hardwoods where it presents difficulties in the efficient utilization of such common indigenous species in Australasia as the eucalypts.

It is now generally agreed that the growth stresses arise from the development of a longitudinal tensile stress and a circumferential compressive stress in the peripheral layer of wood cells as they mature after being laid down from the cambium (Figure 1). These stresses which develop in each outer sheath of cells produce an accumulative strain effect on the existing tracheids composing the internal wood. The aim of this paper is to analyse this process and thereby derive the resultant stress and strain distributions expected in a growing tree. It will be shown that the longitudinal and circumferential stresses at the periphery at each stage of growth produce a radial and tangential tension, together with a substantial longitudinal compression, towards the centre of a tree trunk.

An experimental investigation of the growth stresses and strains involves measuring the strain released in variously-shaped specimens when they are removed from a freshly-felled log. Since this process will cause stress and strain redistributions within the specimen there is often some ambiguity concerning the strains which are being measured. To enable a comparison to be made between the theoretical results and the experimental data an analysis is therefore made of the cutting processes used to produce some of the data. It is shown that the preparation of a thin diametrical plank from a tree results in a uniform extension within the plank of the in-tree longitudinal strain, the radial gradient of this strain remaining unchanged. In contrast, the preparation of a cross-cut disc has a significant effect on the form of the transverse stresses and strains. This effect may be used to resolve the previous conflict reported by several authors between the theoretical radial strain in a tree and the measured radial strain release in a wedge from a cross-cut disc.

KÜBLER (1959 a, b) was the first author to produce a comprehensive theoretical analysis of the mechanics of growth stress development. He compared his theoretical expressions for the in-tree stresses and strains with some experimental data and found good agreement as far as broad qualitative features were concerned. However, the behaviour of the theoretical transverse strains, particularly the radial strain, did not closely match the observed strain-release during the experiments. Since we are dealing with a biological phenomenon with all its inherent variability detailed agreement between theory and experiment can not be expected. Nevertheless, closer agreement between theory and experiment would be expected to lead to a better understanding of the growth stress phenomenon.

KÜBLER investigated the longitudinal and transverse stress developments seperately whereas there will in reality be some interaction between the two because of Poisson ratio effects. GILLIS (1973) investigated a refined model which treated wood as a three-dimensional, isotropic and homogeneous elastic continuum. GILLIS appreciated that to adequately compare theory and experiment a cutting analysis must be performed which models the actual cutting process used in the experiments. However, he found that the predicted radial strain-release during the preparation of a wedge from a cross-cut disc was not in good agreement with the measured strain-releases.

An attempt is made in this paper to produce a model for the development of growth stresses in hardwoods which gives better agreement between theory and experiment. In addition to allowing for the anisotropic character of wood the analysis presented here differs in certain other aspects from that presented by GILLIS. In particular, his assumption that a plane transverse cross-section remains plane during the addition of a new sheath of cells is a consequence of the full stress-equilibrium and strain-compatibility equations used here. Furthermore, the cutting analysis is carried out on a more formal basis.

ANISOTROPIC MODEL OF GROWTH STRESS DEVELOPMENT

The tree trunk is treated as a cylinder of uniform diameter growing outwards by the formation of a sheath of new cells at the periphery at each stage of growth. Once the new cells have been laid down from the cambium and are attached to the existing wood cells at the periphery internal stresses develop in the new cells. The problem is to derive the stress and strain distribution which would develop in the tree under these conditions, treating wood as an elastically homogeneous and cylindrically orthotropic continuum.

Cylindrical coordinates are appropriate and the subscripts 1,2,3 are used to denote the radial, axial and tangential directions respectively. The following notation is used:

r = radial coordinate
z = axial coordinate
θ = tangential coordinate
R = radius of the tree trunk at any given stage of growth
u_i = displacement component
ϵ_i = normal strain component
σ_i = normal stress component
γ_{ij} = shear strain component ($i \neq j$)
τ_{ij} = shear stress component ($i \neq j$)

At each stage of growth the stresses must satisfy the equilibrium conditions for each of the three orthogonal directions. The form of these equations in cylindrical coordinates is given by LEKHNITSKII (1963, equations (1.6)) as:

$$\begin{aligned}
&\frac{\partial \sigma_1}{\partial r} + \frac{\sigma_1 - \sigma_3}{r} + \frac{\partial \tau_{12}}{\partial z} + \frac{1}{r}\frac{\partial \tau_{13}}{\partial \theta} = 0 \\
&\frac{\partial \sigma_2}{\partial z} + \frac{\partial \tau_{12}}{\partial r} + \frac{\tau_{12}}{r} + \frac{1}{r}\frac{\partial \tau_{23}}{\partial \theta} = 0 \qquad (1) \\
&\frac{1}{r}\frac{\partial \sigma_3}{\partial \theta} + \frac{\partial \tau_{13}}{\partial r} + \frac{2\tau_{13}}{r} + \frac{\partial \tau_{23}}{\partial z} = 0
\end{aligned}$$

The effects of gravity have been ignored here. It can be shown that even for very tall trees the stresses and strains arising from the weight of the tree are no more than a few per cent of those arising from the growth phenomenon.

Since we will only consider a portion of the tree trunk remote from the ends and since the loading at the periphery is assumed to be uniform in both the axial and circumferential directions, the stresses and strains are functions of r and R only. Furthermore, the symmetry of the problem implies that the shear stresses τ_{13} and τ_{23} are everywhere zero. The stress equilibrium equations therefore reduce to:

$$r\frac{\partial \sigma_1}{\partial r} + \sigma_1 - \sigma_3 = 0 \qquad (2)$$

$$\frac{\partial}{\partial r}(r\tau_{12}) = 0 \qquad (3)$$

An additional stress condition must be imposed because the stresses are self-generated and this is that there must be no net force on any transverse cross-section of the tree trunk, that is,

$$\int_0^R \sigma_2 r \, dr = 0 \qquad (4)$$

where σ_2 is measured relative to atmospheric pressure and the top end of the tree trunk is exposed to this pressure.

Compatibility conditions which are discussed below must be imposed on the strains. The stresses are therefore expressed in terms of the strains through Hooke's law for an orthotropic elastic material:

$$\sigma_i = c_{i1}\varepsilon_1 + c_{i2}\varepsilon_2 + c_{i3}\varepsilon_3$$
$$\tau_{12} = c_{44}\gamma_{12}\ ,\ \tau_{13} = c_{55}\gamma_{13}\ ,\ \tau_{23} = c_{66}\gamma_{23} \qquad (5)$$

where $c_{ij} = c_{ji}$. This form of Hooke's law for the elastic behaviour of wood is discussed by LEKHNITSKII (1963).

Since the growth strain distribution is generated by an internal mechanism, care must be taken in determining the strain compatibility conditions. At each point within the tree there are two contributions to each component ε_i of the strain. One contribution, denoted by ε_i', is due to internal cell changes which occurred when the cells at the point in question were at the periphery of the tree. This contribution is the source of the growth stress phenomenon. The other contribution, denoted by ε_i^*, is the strain due to the elastic interaction between the successive sheaths of cells. The latter is a function of both r and R whereas ε_i' is independent of R. Only ε_i^* is derivable from a continuous displacement function so that the total strain in any direction is not derivable in this way. A similar problem occurs in the investigation of thermal stresses where ε_i' is due to a temperature distribution throughout the elastic material.

Since ε_i' is independent of R, one can conclude that the change in strain within the tree resulting from the addition of a new sheath of cells at the periphery may be derived from the change in displacement which occurs. This is equivalent to the statement that each $D\varepsilon_i$ may be derived from the Du_i, where D denotes the operator $\partial/\partial R$, in the same way as the strains are derived from the displacements in problems where the stresses and strains arise from the elastic response to external forces only. The strain-compatibility equations for cylindrical co-ordinates in this latter case may be found in LEKHNITSKII(1963, equations (1.10)).

For the present problem these equations must be written in the form:

$$D\varepsilon_1 = \frac{\partial Du_1}{\partial r}\ ,\ D\varepsilon_2 = \frac{\partial Du_2}{\partial z}\ ,\ D\varepsilon_3 = \frac{Du_1}{r}\ ,\ D\gamma_{12} = \frac{\partial Du_2}{\partial r} \qquad (6)$$

There are two additional equations for $D\gamma_{13}$ and $D\gamma_{23}$ but these have been omitted since by symmetry all the quantities involved are zero.

To complete the model the remaining problem is to determine the boundary conditions which are appropriate at each stage of growth. JACOBS (1945) found no systematic variation with tree diameter of the peripheral longitudinal strain in thin diametrical planks. He therefore reached the conclusion that each new sheath of cells develops the same longitudinal tension throughout the life of the tree. A similar conclusion regarding the peripheral circumferential compression is compatible with the more limited evidence available on the transverse stresses. It is therefore assumed that the peripheral longitudinal stress σ_2^o and the peripheral circumferential stress σ_3^o are constant at each stage of growth. It will be shown that this hypothesis is supported by the good agreement between theory and experiment.

Several authors including BOYD (1950c, 1972), have suggested that the source of the peripheral stresses may be strains which arise from changes in cell size that occur as the cells mature, resulting in a contraction in the length of the cells and an expansion in their transverse directions; in particular, BOYD (1972) suggested that these changes are associated with the lignification of the cell walls as they mature. If this were the case one might expect the strain contributions ε_i' discussed earlier to be constant at each stage of growth, since it is reasonable to expect each new sheath of cells to behave in the same way. This author has carried out a growth stress analysis using a model based on the latter hypothesis. However, although the transverse stresses and strains showed the correct qualitative behaviour, the longitudinal strain was constant throughout the tree and was independent of the diameter of the tree. Such behaviour conflicts with available experimental evidence. The experimental support for the hypothesis of constant stresses, and hence constant total strains, at the periphery suggests that the process giving rise to the peripheral stresses is a stress-limited one, at least as far as the longitudinal direction is concerned. It is perhaps of interest to point out that the assumption that σ_2^o and σ_3^o are constant, together with the assumption that the cells behave elastically while they are maturing, implies that ε_1' and ε_3' are constant but that ε_2' is a function of radial position.

With $\sigma_i = \sigma_i\,(r,R)$ and $\tau_{12} = \tau_{12}\,(r,\,R)$, the boundary conditions for the model may be summarised as follows:

$$\sigma_1(R,R)=\sigma_1^o\,,\;\sigma_2(R,R)=\sigma_2^o\,,\;\sigma_3(R,R)=\sigma_3^o\,,\;\tau_{12}(R,R)=0 \qquad (7)$$

All the normal stresses are taken relative to atmospheric pressure so that σ_1^o is due solely to resistance by the bark to radial expansion.

Solution of the Growth Eduations:

Firstly, from equation (3) and the boundary condition on τ_{12} it can be deduced that this shear stress is zero. Thus, the shear strain γ_{12} is also zero and from equations (6),

$$\frac{\partial Du_2}{\partial r} = 0 \qquad (8)$$

It remains to solve equations (2), (4), (5), (6) and (8), with boundary conditions (7), for the normal stresses and strains. The method of solution is to first solve for $D\varepsilon_i$ and $D\sigma_i$, integrate these expressions and use equations (2), (4) and (7) to evaluate the arbitrary functions involved.

Note that equations (7) and (8) imply that $\frac{\partial D\varepsilon_2}{\partial r} = 0$, that is, $D\varepsilon_2$ is a function of R only. It therefore may be expressed in the form:

$$D\varepsilon_2 = DB \tag{9}$$

where B is some function of R only. If equations (5) are substituted in equation (2) and the latter is then differentiated with respect to R, an equation containing the $D\varepsilon_i$ is obtained. This may then be used to give the following equation for Du_1, by using equations (6) and (9),

$$r\frac{\partial^2 Du_1}{\partial r^2} + \frac{\partial Du_1}{\partial r} - p^2 \frac{Du_1}{r} = \frac{c_{23} - c_{12}}{c_{11}} DB \tag{10}$$

where $p^2 = c_{33}/c_{11}$. Equation (10) may be integrated under the restriction that the change in displacement at the axis of the tree trunk as a new layer of cells is added is finite. This gives:

$$Du_1(r,R) = DAr^p + \beta r DB$$

where A is some function of R and β is defined by:

$$\beta = \frac{c_{23} - c_{12}}{c_{11}(1-p^2)} \tag{11}$$

The case $p \neq 1$ is assumed here.

Equations (6) and (9) now lead to:

$$\begin{aligned} D\varepsilon_1 &= pDAr^{p-1} + \beta DB \\ D\varepsilon_2 &= DB \\ D\varepsilon_3 &= DAr^{p-1} + \beta DB \end{aligned} \tag{12}$$

which in turn give:

$$\begin{aligned} D\sigma_1 &= (c_{11}p + c_{13})DAr^{p-1} + \delta DB \\ D\sigma_2 &= (c_{12}p + c_{23})DAr^{p-1} + \gamma DB \\ D\sigma_3 &= p(c_{11}p + c_{13})DAr^{p-1} + \delta DB \end{aligned} \tag{13}$$

where

$$\gamma = \beta c_{12} + \beta c_{23} + c_{22} \tag{14}$$

$$\begin{aligned} \delta &= \beta c_{11} + \beta c_{13} + c_{12} \\ &= \beta c_{13} + \beta c_{33} + c_{23} \end{aligned} \tag{15}$$

the equality of the last two expressions following from equation (11) and the definition of p.

By integrating equations (13) with respect to R and noting that the σ_i are independent of z, explicit expressions may be found for the stresses:

$$\begin{aligned} \sigma_1 &= (c_{11}p + c_{13})A(R)r^{p-1} + \delta B(R) + f_1(r) \\ \sigma_2 &= (c_{12}p + c_{23})A(R)r^{p-1} + \gamma B(R) + f_2(r) \\ \sigma_3 &= p(c_{11}p + c_{13})A(R)r^{p-1} + \delta B(R) + f_3(r) \end{aligned} \tag{16}$$

where the f_i are functions yet to be determined. They can be evaluated by utilising equations (7) which give:

$$\sigma_1^\circ = (c_{11}p + c_{13})A(R)R^{p-1} + \delta B(R) + f_1(R) \quad (17)$$

$$\sigma_2^\circ = (c_{12}p + c_{23})A(R)R^{p-1} + \gamma B(R) + f_2(R) \quad (18)$$

$$\sigma_3^\circ = p(c_{11}p + c_{13})A(R)R^{p-1} + \delta B(R) + f_3(R) \quad (19)$$

Use D to operate on equation (4), then:

$$\int_0^R D\sigma_2 r\,dr = -R\sigma_2^\circ \quad (20)$$

This gives:

$$\frac{c_{12}p + c_{23}}{1+p} R^{p-1} DA + \tfrac{1}{2}\gamma\, DB = -\frac{\sigma_2^\circ}{R} \quad (21)$$

Furthermore, if equations (16) are substituted into equation (2), we find:

$$r f_1'(r) + f_1(r) - f_3(r) = 0 \quad (22)$$

Equations (17), (18), (19), (21), and (22) allow us to determine the five unknown functions A, B, f_1, f_2 and f_3.

By replacing R with r in equations (17) and (19), $f_1(r)$ and $f_3(r)$ may be determined in terms of A(r) and B(r). These expressions may then be substituted into equation (22). Transforming back to the independent variable R, we find that this leads to:

$$(c_{11}p + c_{13})R^{p-1} DA + \delta\, DB = \frac{\sigma_1^\circ - \sigma_3^\circ}{R} \quad (23)$$

The integrated solution of equations (21) and (23) gives

$$B(R) = -\varepsilon \ln R + k_1 \quad (24)$$

$$A(R) = -\frac{(1+p)(\sigma_2^\circ - \frac{1}{2}\gamma\varepsilon)}{(1-p)(c_{12}p + c_{23})} R^{1-p} + k_2 \quad (25)$$

where k_1, k_2 are constants and:

$$\varepsilon = \frac{\alpha\sigma_2^\circ - (\sigma_3^\circ - \sigma_1^\circ)}{\frac{1}{2}\alpha\gamma - \delta} \quad (26)$$

and

$$\alpha = \frac{(1+p)(c_{11}p + c_{13})}{(c_{12}p + c_{23})} \quad (27)$$

$f_i(R)$, and hence $f_i(r)$, may now be determined from one of equations (17) - (19). Substitution of the $f_i(r)$ into equations (16) then leads to the final expressions for the growth stresses:

$$\sigma_1 = \sigma_1^\circ + \frac{1}{p-1}(\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon)\left[\left(\tfrac{r}{R}\right)^{p-1} - 1\right] + \delta\varepsilon \ln\tfrac{r}{R} \quad (28)$$

$$\sigma_2 = \sigma_2^\circ + \frac{p+1}{p-1}(\sigma_2^\circ - \tfrac{1}{2}\gamma\varepsilon)\left[\left(\tfrac{r}{R}\right)^{p-1} - 1\right] + \gamma\varepsilon \ln\tfrac{r}{R} \quad (29)$$

$$\sigma_3 = \sigma_3 + \frac{p}{p-1}(\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon)\left[\left(\tfrac{r}{R}\right)^{p-1} - 1\right] + \delta\varepsilon \ln\tfrac{r}{R} \quad (30)$$

Here the relationship $\frac{(1+\rho)(\sigma_2^\circ - \frac{1}{2}\gamma\varepsilon)}{(c_{12}\rho + c_{23})} = \frac{\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon}{c_{11}\rho + c_{13}}$ from equations (26) and (27) has been used to express the coefficient of $[(\frac{r}{R})^{\rho-1} - 1]$ in σ_1 and σ_3 in a more appropriate form.

The corresponding strains may be most readily derived by integrating equations (12) and by using the peripheral strains $\varepsilon_i^\circ = \varepsilon_i(R,R)$ to determine the arbitrary functions introduced by the integration in much the same way as the f_i were determined. The following expressions for the strains may then be deduced:

$$\varepsilon_1 = \varepsilon_1^\circ + \frac{\rho(\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon)}{(\rho-1)(c_{11}\rho + c_{13})}\left[\left(\frac{r}{R}\right)^{\rho-1} - 1\right] + \beta\varepsilon \ln\frac{r}{R} \qquad (31)$$

$$\varepsilon_2 = \varepsilon_2^\circ + \varepsilon \ln\frac{r}{R} \qquad (32)$$

$$\varepsilon_3 = \varepsilon_3^\circ + \frac{(\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon)}{(\rho-1)(c_{11}\rho + c_{13})}\left[\left(\frac{r}{R}\right)^{\rho-1} - 1\right] + \beta\varepsilon \ln\frac{r}{R} \qquad (33)$$

The σ_i° and ε_i° appearing in these equations are, of course, related through equation (5).

Since there are no shear stress and strains, equations (28) - (33) give a complete description of the stress conditions within the tree.

COMPARISON OF GROWTH STRESS MODELS

In this section the results of the anisotropic model are compared with those of two previously published models by KUBLER (1959 a,b) and GILLIS (1973) of the development of growth stresses. For this purpose the parameters appearing in the models must be given relevant values, in particular, ε_2° and ε_3°. JACOBS (1945) reports a tensile strain of the order of 1×10^{-3} in the longitudinal direction at the periphery of specimens of Eucalyptus gigantea, while BOYD(1950 a) reports a peripheral compressive strain of the same order for the circumferential direction in several species of Eucalyptus. For comparison purposes, it will be adequate to take $\varepsilon_2^\circ = -\varepsilon_3^\circ = \varepsilon^\circ$. Furthermore, the relatively small resistance by the bark to radial expansion is neglected and consequently $\sigma_1^\circ = 0$.

There are six independent elastic constants for an orthotropic material as far as the normal stresses and strains are concerned. To enable a comparison to be made between the anisotropic model and the other two models the following set of values, based on some tabulated values for hardwoods, are taken for the elastic constants:

$$E_1 = 0.1\,E_2\,,\ E_2 \text{ unspecified},\ E_3 = 0.5\,E_1$$
$$\nu_{21} = 0.5\,,\ \nu_{23} = 0.5\,,\ \nu_{13} = 0.7 \qquad (34)$$

Here ν_{ij} is a Poisson's ratio, being the ratio of contraction in the j-th direction to extension in the i-th direction when a narrow cylinder of material with its axis in the i-th direction is stretched in that direction. The stiffnesses c_{ij} are therefore given by:

$$c_{11} = 0.141\,E_2\,,\ c_{22} = 1.08\,E_2\,,\ c_{33} = 0.070\,E_2$$
$$c_{12} = 0.096\,E_2\,,\ c_{13} = 0.052\,E_2\,,\ c_{23} = 0.061\,E_2 \qquad (35)$$

while the elastic parameters appearing in the anisotropic analysis reduce to:

$$p = 0.703,\ \alpha = 2,\ \beta = -0.5,\ \gamma = E_2$$
$$\delta = 0,\ \varepsilon = [2\sigma_2^\circ - \sigma_3^\circ]/E_2 = 2\varepsilon_2^\circ \tag{36}$$

The particularly simple form of $\alpha, \beta, \gamma, \delta$ and ε is a consequence of the equality of ν_{21} and ν_{23}. In fact it can be shown that if $\nu_{21} = \nu_{23} = \nu_0$ then:

$$\alpha = \frac{1}{\nu_0},\ \beta = -\nu_0,\ \gamma = E_2,\ \delta = 0 \text{ and } \varepsilon = 2[\sigma_2^\circ - \nu_0(\sigma_3^\circ - \sigma_1^\circ)]/E_2.$$

The anisotropic growth strains given by equations (31) - (33), under the above assumptions concerning the values of the parameters appearing in the expressions, are plotted in a normalised form in Figure 2. The corresponding normalised stresses are shown in Figure 3. The stresses are normalized using the peripheral strains rather than the peripheral stresses since it is the former which are directly amenable to experimental measurements. The theoretical stress behaviour is in qualitative agreement with observations made on trees, from which it is known that the longitudinal stress changes from a tension near the periphery to a substantial compression near the centre of the tree trunk ($E_2 \sim 10^{10}$ N.m $^{-2}$ and $\varepsilon_2^\circ \sim 10^{-3}$), while the radial and tangential stresses are more modest and develop into a tension near the centre. Note also that the similar behaviour of the solid curves of Figure 2 and 3 implies that the longitudinal stress is practically given by the longitudinal strain multiplied by the longitudinal Young's modulus.

The results of two other models analysing the development of growth stresses have also been plotted in Figures 2 and 3 under the above assumption concerning the peripheral strains. KUBLER (1959 a, b) treated the transverse and longitudinal problems separately thereby preventing interaction between them. His expressions for the growth stresses and strains are:

$$\sigma_1 = E_3\varepsilon_3^\circ \ln\frac{r}{R}$$
$$\sigma_2 = E_2\varepsilon_2^\circ\left(1 + 2\ln\frac{r}{R}\right)$$
$$\sigma_3 = E_3\varepsilon_3^\circ\left(1 + \ln\frac{r}{R}\right)$$
$$\varepsilon_1 = (E_3/E_1)\varepsilon_3^\circ \ln\frac{r}{R}$$
$$\varepsilon_2 = \varepsilon_2^\circ\left(1 + 2\ln\frac{r}{R}\right)$$
$$\varepsilon_3 = \varepsilon_3^\circ\left(1 + \ln\frac{r}{R}\right)$$

GILLIS (1973) used a three-dimensional model but for simplicity he assumed that wood was elastically isotropic. His expressions for the growth stresses and strains may be derived from the anisotropic expressions by replacing the stiffnesses c_{ij} with their isotropic values. The anisotropic expressions are indeterminate when $c_{11} = c_{33}$, that is p = 1, so that the limit as p tends to 1 must be used. To plot the isotropic stress expressions some allowance for the anisotropic character of wood may be made by replacing the isotropic Young's modulus E by E_3 in the expressions for σ_1 and σ_3 and by E_2 in the expression for σ_2. It is then found that the isotropic stress distribution is a multiple $\frac{1}{1+\nu}$ of the stress distribution of KUBLER, ν being the isotropic Poisson's ratio. Consequently, only the latter distribution has been plotted in Figure 3.

It is clear from Figure 2 that allowing the transverse strains to affect the longitudinal strain through Poisson ratio effects does not alter the longitudinal strain significantly since the latter has practically the same variation with r/R in all three models. This is what one might expect since the Poisson ratios, ν_{12} and ν_{32}, governing the Poisson effects in this case are small. For example, with the values of the elastic constants given in equations (34):

$$\nu_{12} = \nu_{21}\, E_1/E_2 = 0.05\,, \quad \nu_{32} = \nu_{23}\, E_3/E_2 = 0.025$$

The relationship between the Poisson ratios which is used here may be found, for example, in LEKHNITSKII (1963). On the other hand it appears that the longitudinal strain has a significant effect on the transverse strains. Again this is what might be expected since the relevant Poisson ratios are now ν_{21} and ν_{23} and these are relatively large.

Because of the excellent agreement between the longitudinal strains of all three models there is a corresponding agreement between the longitudinal stresses. This is illustrated in Figure 3 where the longitudinal stress of KUBLER (1959 b) and those of the anisotropic model are so close that the resolution of this figure is not sufficient to clearly distinguish them. This good agreement is a consequence of the fact that the relatively large longitudinal Young's modulus results in the longitudinal strain being the dominant contribution to the longitudinal stress in a three-dimensional model. For other values of the elastic onstants for which $\nu_{21} \neq \nu_{23}$ the agreement between the longitudinal components is still very good but not quite as striking as that shown in Figures 2 and 3.

COMPARISON OF THEORY AND EXPERIMENT

Several authors have published experimental work of a quantitative nature on longitudinal growth stresses and strains. These include JACOBS (1945), BOYD (1950 a), KUBLER (1959 b) and GIORDANO et al (1969). Experimental data on the transverse growth stresses and strains is more sparse and most of the quantitative data available is due to BOYD (1950 a) and KUBLER (1959 a). Because of the "built-in" nature of the growth stresses and strains the experimental work does not involve direct measurements on a growing tree but rather it involves measuring the strains released as a log is cut up in various ways. This often leads to some ambiguity of the meaning of the strains measured. For example, the experimental technique used to deduce the radial strain existing in a tree involves the initial preparation of a cross-cut disc. This process will affect the internal strain distribution through Poisson ratio effects because practically all of the longitudinal stress will be released in the disc. The subsequent radial strain-release which is measured in a wedge cut from the disc will therefore not correspond exactly to the in-tree radial strain. Thus one must proceed with caution when attempting to compare measured strain releases with the strains existing in a growing tree as predicted by theory.

To make an unambiguous comparison between theory and experiment a theoretical cutting analysis simulating the actual experimental techniques could be made to deduce the strain-release predicted by the theoretical model. Alternatively, one could deduce the in-tree stresses and strains for direct comparison with the theoretical distributions by working back from the experimental data available. However it would still be necessary to invoke some elastic model for wood for this so that the two approaches to comparing theory and experiment are essentially equivalent. The former approach will be used here.

Experimental measurements of the growth strains are performed on freshly-felled logs. The question therefore arises as to whether the stress release at the cut faces results in any significant change in the growth stresses and strains. The experimental work of BOYD (1950 a) shows that such changes are confined to the end regions of a log. For a two-feet diameter log of <u>Eucalyptus regnans</u> there is no significant strain-release at distances greater than three feet from each end.

cf. St Venant [illegible]

Transverse Strains

Although previously published models have produced good agreement with the longitudinal strain observed during experiments, this has not been the case with transverse strains, particularly the radial strain. The experimental evidence of BOYD (1950 a) and KUBLER (1959 a) suggests that the radial strain released during preparation of a wedge from a cross-cut disc varies very little with respect to radial position towards the outer portion of the tree trunk. It is clear from Figure 2 that the theoretical behaviour of the radial strain in the tree does not display this feature. The question then remains whether the cutting process itself produces this behaviour of the radial strain.

GILLIS (1973) analysed the cutting process using an isotropic model and found significant alterations in the transverse stresses and strains during the preparation of a cross-cut disc. However, his predicted radial strain-release still does not agree well with the experimental evidence available. GILLIS speculated that a full anisotropic analysis might overcome this discrepancy between theory and experiment. As shown by Figure 2 this has not been achieved by an alteration of the in-tree radial strain for this has an even steeper gradient as one moves in from the periphery than that of the previous models. An anisotropic cutting analysis will now be presented which shows that the preparation of a cross-cut disc during the experiments leads to the observed behaviour of the radial strain.

(i) Analysis of strain in a cross-cut disc

A cross-cut disc is produced by cuts across two parallel planes normal to the stem axis. The theoretical stress and strain distributions are to first order independent of the actual cutting technique used to prepare the specimen because the governing equations are linear. In practice the existence of some irreversible phenomenon, such as the formation of splits, may lead to some dependence on the cutting process. In the experiments care is taken to reduce the extent of the heart shakes, or radial splits, which often appear at a cross-cut face. BOYD (1950 a) describes the actual cutting process in detail.

The preparation of a cross-cut disc from a log results in the longitudinal stress being released at each face of the disc. This leads to a redistribution of the stresses and strains within the specimen. If the superscript t is now used to distinguish the in-tree stresses and stains (equations (28) - (30) and (31) - (33)), the stresses in the disc, denoted by the same symbols as used previously for within the tree, must satisfy the equilibrium conditions given in equations (1). As before these equations may be simplified because there is circular symmetry. Thus, τ_{13} and τ_{23} are everywhere zero and there is no dependence on θ. The disc stresses and strains must also satisfy Hooke's law (equations (5)).

For reasons presented earlier it is the change in strains produced by the cross-cutting which must satisfy compatibility conditions. Let the operator Δ be defined by:

$$\Delta e = e - e^{(t)}$$

where e is some elastic quantity, then the compatibility conditions are:

$$\Delta\varepsilon_1 = \frac{\partial \Delta u_1}{\partial r}, \; \Delta\varepsilon_2 = \frac{\partial \Delta u_2}{\partial z}, \; \Delta\varepsilon_3 = \frac{\Delta u_1}{r}$$
$$\Delta\gamma_{12} = \frac{\partial \Delta u_1}{\partial z} + \frac{\partial \Delta u_2}{\partial r} \qquad (37)$$

Note that these equations are analogous to equations (6) except for an additional term in the shear strain expression because now the strain may depend on z and hence we can no longer conclude that Δu_1 must be independent of z.

Suppose that the disc is of thickness 2d and that the cross-cut faces are at z = ± d. The boundary conditions may then be expressed as:

$$\tau_{12} = 0 \ , \ \sigma_1 = \sigma_1^{\circ} \quad \text{at} \quad r = R, \ -d \leq z \leq d$$

$$\tau_{12} = 0 \ , \ \sigma_2 = 0 \quad \text{at} \quad z = \pm d, \ 0 \leq r \leq R$$

To solve the equations under these boundary conditions one first solves for the changes in the stresses and strains. By subtracting the appropriate equations it may be shown that these changes satisfy equations (37) together with the following:

$$r\frac{\partial \Delta\sigma_1}{\partial r} + \Delta\sigma_1 - \Delta\sigma_3 + r\frac{\partial \Delta\tau_{12}}{\partial z} = 0 \qquad (38)$$

$$r\frac{\partial \Delta\sigma_2}{\partial z} + \frac{\partial}{\partial r}(r\,\Delta\tau_{12}) = 0 \qquad (39)$$

$$\Delta\sigma_i = c_{i1}\Delta\varepsilon_1 + c_{i2}\Delta\varepsilon_2 + c_{i3}\Delta\varepsilon_3 \ , \ i = 1,2,3 \qquad (40)$$

$$\Delta\tau_{12} = c_{44}\Delta\gamma_{12} \qquad (41)$$

$$\Delta\tau_{12} = 0 \ , \ \Delta\sigma_1 = 0 \ , \quad \text{at } r = R, \ -d \leq z \leq d \qquad (42)$$

$$\Delta\tau_{12} = 0 \ , \ \Delta\sigma_2 = -\sigma_2^{(t)}\left(\tfrac{r}{R}\right), \ \text{at } z = \pm d, \ 0 \leq r \leq R \qquad (43)$$

In their full generality these equations are difficult to solve. Fortunately, a solution can readily be found for the limiting case of a thin disc, which is appropriate to the situation being examined, that is, it is assumed that the parameter $\lambda = 2R/2d =$ (diameter of disc)/ (thickness of disc) is much greater than unity. Under this condition the absence of shear at each cross-cut face is assumed to prevail throughout the disc, that is, $\Delta\tau_{12} = 0$ everywhere. Using equations (39) and (43) this then implies that

$\Delta\sigma_2 = -\sigma_2^{(t)}\left(\tfrac{r}{R}\right)$ throughout the disc, so that:

$$c_{22}\Delta\varepsilon_2 = -\sigma_2^{(t)} - c_{12}\Delta\varepsilon_1 - c_{23}\Delta\varepsilon_3 \qquad (44)$$

Utilising equations (37), (40) and (44), equation (38) leads to the following equation for Δu_1:

$$r\frac{\partial^2 \Delta u_1}{\partial r^2} + \frac{\partial \Delta u_1}{\partial r} - q^2\frac{\Delta u_1}{r} = \frac{1}{c_{22}c_{11} - c_{12}^2}\left[c_{12}\, r\frac{\partial \sigma_2^{(t)}}{\partial r} - (c_{23} - c_{12})\sigma_2^{(t)}\right] \qquad (45)$$

where $$q^2 = \frac{c_{33}c_{22} - c_{23}^2}{c_{22}c_{11} - c_{12}^2} = \frac{E_3}{E_1} \approx p^2$$

The solution of equation (45) has the following form:

$$\Delta u_1 = Ar\ln\tfrac{r}{R} + Br + Cr\left(\tfrac{r}{R}\right)^{p-1} + Dr\left(\tfrac{r}{R}\right)^{q-1} \qquad (46)$$

The constants A, B and C may be evaluated directly from equation (45) by comparing like terms in r, and D may be evaluated by utilising the remaining boundary condition, $\Delta\sigma_1 = 0$ at r = R. It may be shown that A, B, C and D are given by the following equations:

$$F = C_{22}C_{11} - C_{12}^2$$

$$A = -\frac{\gamma_\varepsilon (C_{23} - C_{12})}{F(1-q^2)}$$

$$B.F(1-q^2) = -2A.F + C_{12}\gamma_\varepsilon + \frac{C_{23}-C_{12}}{p-1}\left[2\sigma_2^0 - \frac{p+1}{2}\gamma_\varepsilon\right] \tag{47}$$

$$C = \frac{(p+1)(\sigma_2^0 - \frac{1}{2}\gamma_\varepsilon)(C_{12}p - C_{23})}{(p-1)(p^2-q^2)F}$$

$$D(q+\nu_{31}) = C_{12}\sigma_2^0/F - A$$

Once A, B, C and D have been evaluated, the change in transverse strains may be calculated from the following expressions which are derived from equations (37) and (46):

$$\Delta\varepsilon_1 = A + B + A\ln\frac{r}{R} + pC\left(\frac{r}{R}\right)^{p-1} + qD\left(\frac{r}{R}\right)^{q-1}$$
$$\Delta\varepsilon_3 = B + A\ln\frac{r}{R} + C\left(\frac{r}{R}\right)^{p-1} + D\left(\frac{r}{R}\right)^{q-1} \tag{48}$$

$\Delta\varepsilon_2$ may be calculated from equation (44). The change in stresses may then be evaluated by means of equations (40).

The above solution for the change in strains and stresses satisfies all of the necessary equations except the shear strain compatibility condition in equation (37). For $\Delta\tau_{12} = 0$ everywhere implies $\Delta\gamma_{12}$ must also be zero everywhere, but $\Delta\gamma_{12} = \frac{\partial \Delta u_2}{\partial r}$ since Δu_1 is independent of z. Furthermore, since $\Delta\varepsilon_2$ is a function of r so is Δu_2 and hence $\frac{\partial \Delta u_2}{\partial r}$ is not identically zero. However, if Δu_1 and Δu_2 are replaced by their dimensionless forms $\Delta u_1' = \Delta u_1/\varepsilon_3^0 R$ and $\Delta u_2' = \Delta u_2/\varepsilon_2^0 d$, and r and z are scaled by 1/R and 1/d respectively, then it can be deduced that:

$$\Delta\gamma_{12} = \varepsilon_2^0/\lambda \quad \times \quad \text{(a term of order unity or less}$$

Thus, the shear strain compatibility condition is satisfied in the limit as λ becomes very large. In particular the above solution is a good approximation to the exact solution in the case of a thin disc where $\lambda \gg 1$.

The constants A, B, C, D and q have been evaluated for the values of the elastic constants given in equations (34) and for

$\sigma_1^0 = 0$, $\varepsilon_2^0 = -\varepsilon_3^0 = \varepsilon^0$. This gave:

$$A = 1.000\,\varepsilon^0,\ B = -0.843\,\varepsilon^0,\ C = -0.564\,\varepsilon^0$$
$$D = 1.322\,\varepsilon^0,\ q = 0.707 \tag{49}$$

The changes in stresses and strains were then evaluated and these were used to calculate the actual stresses and strains in the disc. The results obtained are plotted in Figures 4 and 5.

Figure 2, 3, 4 and 5 which are drawn to the same scale show that the preparation of a cross-cut disc has a dramatic effect on the internal transverse stresses and strains. Towards the centre of the tree trunk the transverse strains are significantly reduced whereas the transverse stresses are more than doubled. This contrary behaviour occurs because the reduction in the longitudinal strain practically nullifies its previous role as a substantial counteracting contribution to the transverse stresses. Towards the outer portion of the tree the transverse stresses and strains have all increased in magnitude; the increases in the radial strain and tangential stress are particularly substantial.

It should be observed that although a significant alteration of the transverse stresses and strains takes place during the preparation of a cross-cut disc, the peripheral circumferential strain is practically unaltered. BOYD (1950 a) did in fact observe this behaviour during his experiments when he found no significant change in the diameter of a log as a disc was cut from it. Since ε_3^0 is known to be of the order of 10^{-3} the theoretical change in the circumferential strain shown in Figure 4 is less than 10^{-4}. The quoted accuracy of the strain gauge used by BOYD implies that such a strain change would not be detected. BOYD erroneously concluded that this lack of detectable change in the circumferential strain implied that the transverse stresses were not altered substantially during the removal of a cross-cut disc from a log.

It should also be observed that cross-cutting has caused the tangential tensile stress near the pith to more than double. A similar behaviour would be expected at each face of a single cross-cut through a log and this could be responsible for initiating the phenomenon of heart shakes (BOYD, 1950 b) whereby radial splits propagate out from the pith during cross-cutting. Previous analyses of this phenomenon using simpler elastic models for wood, such as that presented by GILLIS and BURDEN (1972) have indicated that the increase in the tangential stress during cross-cutting would be somewhat smaller and therefore not sufficient to cause tangential tensile failure.

This increase in the tangential tensile stress is a direct result of the release of longitudinal stress at a cross-cut face. There is a further contribution to the increase in the transverse stresses which arises in the single cross-cut situation but not in the double cross-cut, thin disc situation. This is due to the variation of the longitudinal stress from zero at the cross-cut face to the in-tree value of the stress well in from the face. This variation generates a shear stress τ_{12} (see equation (39)) which, near the cross-cut face, increases in magnitude away from the face. The shear stress in turn makes a contribution to the transverse stresses (see equation (38)) in the direction of increasing the tangential tensile stress. This would make the heart shake phenomenon even more marked.

A shear stress τ_{12} is not present in the thin disc situation because the longitudinal stress is released throughout the disc. This is also why the technique used by BOYD (1950 a) to produce a cross-cut disc by successive cuts which approach the desired specimen significantly reduces the extent of any heart shakes in the final disc. By the time the last cross-cuts are made, the longitudinal stress has again been released throughout the specimen.

(ii) <u>Analysis of strain in a wedge cut from a cross-cut disc</u>

During experiments to determine the radial strain the strain release is measured along a radial line in a thin wedge cut from a cross-cut disc. To analyse this process one need only observe that the appropriate radial stress-balance equation is the simplified form given in equation (2) where only the terms containing σ_1 and σ_3 appear. Thus, if the wedge is sufficiently thin so that the stress condition $\sigma_3 = o$ prevailing at the faces of the new cuts may be assumed to hold throughout the wedge, the stress-balance equation implies that σ_1 is also zero everywhere in the wedge. Thus all the stresses, and consequently all the strains, have been released. The measured strain-release between any two points in a radial line must therefore correspond directly to the radial strain existing in the cross-cut disc but of course it will be of opposite sign. BOYD (1950 a) confirmed experimentally that it is indeed the case that all the radial strain is released when a wedge is cut from a cross-cut disc because when he further subdivided a wedge he found no significant additional strain-release.

We are now in a position to resolve the previous conflict between theory and experiment with regard to the radial strain-release. Figure 4 shows that the theoretical radial strain-release in a wedge has only a modest variation with radial position towards the outer portion of the tree trunk in agreement with the experimental evidence. For comparison purposes the measurements of KUBLER (1959 a) for a cross-cut disc taken from a particular species of northern beech are also plotted in Figure 4. Fortunately KUBLER also gives the circumferential strain near the periphery for the specimen he used so that the actual strain-release values, with a change in sign, may be plotted in their normalised form. These experimental points show remarkable agreement with the theory when one considers that the elastic parameters used in the theoretical curve are merely typical values and they are almost certain to differ significantly from the actual values appropriate to the experimental specimen. However, this is in accordance with the observation that the theoretical normalised curves for a cross-cut disc are found to be somewhat insensitive to changes in the values of the parameters. The only significant disagreement between the measurements of KUBLER and the theoretical curve is where the experimental values fall off before increasing again near the centre. This anomalous behaviour is presumably due to some peculiarity during the history of the specimen. The results of BOYD (1950 a) for a Eucalyptus regnans tree are also shown in Figure 4. BOYD does not present the peripheral circumferential strain for his specimen and this is required to normalise the strains correctly. The values have therefore been scaled by a somewhat arbitrary factor. These results also show good agreement with the theory except for a reduction in the radial strain near the pith. The similarilty between this anomalous behaviour in the results of BOYD and of KUBLER may be coincidental or it may be due to the minor heart shakes which are associated with the experimental technique used to prepare the cross-cut discs.

KUBLER (1959 a) also presents some data on the radial variation of the tangential strain-release. Unfortunately he does not explain the cutting technique used to release the tangential strain. However, assuming that his technique does indeed measure all of the tangential strain present in a cross-cut disc and only this strain his results can be compared directly with the normalized curve in Figure 4. There is good agreement between his results and the theory in the outer portion of the tree but his measurements indicate that the tangential strain changes from a contraction to an extension further out from the centre than that which occurs in the theory.

The tangential and radial strain-release measurements made by KUBLER and plotted in Figure 4 were both made on the same log specimen of a northern beech.

Longitudinal Strains

Almost all the experimental work (JACOBS, 1945; KUBLER, 1959 b; GIORDANO 1969) on longitudinal growth strains has been carried out on diametrical planks, that is planks parallel to the axis of a tree trunk which have a width equal to the diameter of the trunk and which have the pith at their centre. The redistribution of stresses and strains during the preparation of such a plank from a log will be investigated in this section. It will be shown that the preparation of a diametrical plank results in a uniform change in the longitudinal strain and that the theoretical variation with tree radius of the peripheral longitudinal strain gradient in a diametrical plank fits the data of JACOBS (1945) very well. Furthermore, it will be shown that the theory predicts that the radius of curvature of a thin peripheral strip cut from a diametrical plank is proportional to the radius of the tree trunk, again giving agreement with some results published by JACOBS (1965). However the partial solution used in the cutting analysis is not capable of predicting the length change which will occur when a thin peripheral strip is removed from the plank.

(i) Analysis of strain in a thin diametrical plank

The stresses within a diametrical plank must satisfy the equilibrium conditions given in equations (1) together with the condition that there be no net force on any cross-section perpendicular to the axis of the plank. The stresses must also satisfy appropriate boundary conditions which express the fact that there is no shear stress along a bounding surface and that there is no stress normal to such a surface.

The strains within the plank are related to the stress through Hooke's law (equation (5)) and they must also satisfy compatibility conditions. As before it is the change in strains which are related to the change in displacements because of the presence of initial strains which were generated in the tree by an internal mechanism and not be elastic interaction. Equations (1.10) of LEKHNITSKII (1963) indicate that the strain compatibility conditions may be written as

$$\Delta\varepsilon_1 = \frac{\partial \Delta u_1}{\partial r}, \quad \Delta\varepsilon_2 = \frac{\partial \Delta u_2}{\partial z}, \quad \Delta\varepsilon_3 = \frac{\Delta u_1}{r} + \frac{1}{r}\frac{\partial \Delta u_3}{\partial \theta}$$
$$\Delta\gamma_{13} = \frac{1}{r}\frac{\partial \Delta u_1}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{\Delta u_3}{r}\right), \quad \Delta\gamma_{12} = \frac{\partial \Delta u_1}{\partial z} + \frac{\partial \Delta u_2}{\partial r} \qquad (50)$$
$$\Delta\gamma_{23} = \frac{1}{r}\frac{\partial \Delta u_2}{\partial \theta} + \frac{\partial \Delta u_3}{\partial z}$$

This author has not been able to obtain a plausible full solution to the system of equations described above for the stresses and strains within a diametrical plank. It may be shown that the simple uniaxial stress distribution where the only stress present is parallel to the axis of the plank is a solution of the equations of the problem under the restriction $\nu_{21} = \nu_{23}$ but is not a plausible solution. This is because under the above conditions the strain compatibility equations imply that the tangential displacement within the plank has the form $u_3 = k\,\theta\, r$, where k is a non-zero constant which depends upon the peripheral strains in the growing tree. The uniaxial stress solution therefore gives rise to a multi-valued function of θ for the tangential displacement, indicating that the solution is not a useful one. However it will be shown that the form of the longitudinal strain for the true solution of the diametrical plank problem may be deduced and that this allows some comparison between theory and experiment.

It is assumed that the plank is long and that the portion of the plank being examined is well away from the ends. In this situation the stresses and strains will not depend on z. Let $D = \Delta\varepsilon_2$ where Δ is an operator denoting the difference between the in-plank and in-tree values of a quantity. D is independent of z so that by integration of the second of equations (50) it follows that:

$$\Delta u_2 = D(r,\theta) z + W(r,\theta) \qquad (51)$$

By symmetry τ_{23} must be zero. It then follows from the second of equations (1) and the boundary condition $\tau_{12} = 0$ on the curved sides of the plank that the shear stress τ_{12} is also zero throughout. Thus, from Hooke's law and the fact that there is no shear strain in the tree, it follows that $\Delta\gamma_{12} = \Delta\gamma_{23} = 0$. The last two of equations (50) may therefore be integrated with respect to z to give:

$$\begin{aligned} \Delta u_1 &= -\frac{1}{2}\frac{\partial D}{\partial r} z^2 - \frac{\partial W}{\partial r} z + U \\ \Delta u_3 &= -\frac{1}{2}\frac{1}{r}\frac{\partial D}{\partial \theta} z^2 - \frac{1}{r}\frac{\partial W}{\partial \theta} z + V \end{aligned} \qquad (52)$$

Here the arbitrary functions U and V are functions of r and θ. Equations (51) and (52) may now be used to solve for $\Delta\varepsilon_1$, $\Delta\varepsilon_3$ and $\Delta\gamma_{13}$ by making the appropriate substitutions in equations (50). Thus:

$$\begin{aligned} \Delta\varepsilon_1 &= -\frac{1}{2}\frac{\partial^2 D}{\partial r^2} z^2 - \frac{\partial^2 W}{\partial r^2} z + \frac{\partial U}{\partial r} \\ \Delta\varepsilon_3 &= -\frac{1}{2}\frac{1}{r}\left[\frac{\partial D}{\partial r} + \frac{1}{r}\frac{\partial^2 D}{\partial \theta^2}\right] z^2 - \frac{1}{r}\left[\frac{\partial W}{\partial r} + \frac{1}{r}\frac{\partial^2 W}{\partial \theta^2}\right] z + U/r + \frac{1}{r}\frac{\partial V}{\partial \theta} \\ \Delta\gamma_{13} &= -\frac{1}{2}\frac{1}{r}\left[\frac{\partial^2 D}{\partial r \partial \theta} + r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial D}{\partial \theta}\right)\right] - \frac{1}{r}\left[\frac{\partial^2 W}{\partial r \partial \theta} + r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial W}{\partial \theta}\right)\right] z + \frac{1}{r}\frac{\partial U}{\partial \theta} + r\frac{\partial}{\partial r}(V/r) \end{aligned} \qquad (53)$$

However $\Delta\varepsilon_1$, $\Delta\varepsilon_3$ and $\Delta\gamma_{13}$ must be independent of z.

Equations (53) therefore imply that:

$$\frac{\partial^2 D}{\partial r^2} = 0, \quad \frac{\partial D}{\partial r} + \frac{1}{r}\frac{\partial^2 D}{\partial \theta^2} = 0, \quad \frac{\partial^2}{\partial r \partial \theta}\left(\frac{D}{r}\right) = 0 \qquad (54)$$

The general solution of these equations is:

$$D(r,\theta) = rA\sin\theta + rB\cos\theta + C$$

But from symmetry ε_2 must have the same value at conjugate points with respect to the two principal axes of any transverse cross-section of the plank, that is:

$$\varepsilon_2(r,\theta) = \varepsilon_2(r,\pi-\theta) = \varepsilon_2(r,\pi+\theta) = \varepsilon_2(r,-\theta)$$

Since the in-tree longitudinal strain $\varepsilon_2^{(t)}$ is independent of θ it follows that $D = \Delta\varepsilon_2 = \varepsilon_2 - \varepsilon_2^{(t)}$ also possesses the above symmetry properties. The constants A and B in the expression for D must therefore be zero, that is:

$$\Delta \varepsilon_2 = D(r,\theta) = C \quad \text{, a constant,}$$

and the longitudinal strain within the plank is given by:

$$\varepsilon_2^{(P)} = \Delta \varepsilon_2 + \varepsilon_2^{(t)} = (C + \varepsilon_2^{a}) + \varepsilon \ln \frac{r}{R} \tag{55}$$

Unfortunately to evaluate the constant C one must use the solution for the transverse stresses and this author has not been able to obtain these. However since most of the experimental measurements of JACOBS (1945) may be shown to relate to the radial gradient of the longitudinal strain some comparison between theory and experiment may still be made. To achieve this the effect of removing a thin peripheral strip from each side of the plank must be analysed.

(ii) <u>Analysis of strain in thin peripheral strips cut from a diametrical plank.</u>

Suppose that the diametrical plank is of thickness 2h and that a peripheral strip of width 2t is removed from each side of the plank. It is assumed that the peripheral strips are thin in both transverse directions so that $\frac{h}{R} \ll 1$ and $\frac{t}{R} \ll 1$ where R is the radius of the tree trunk, that is 2R is the width of the diametrical plank.

The stresses and strains within a long strip must satisfy the same equations and boundary conditions as those described earlier for the diametrical plank. In particular, equations (50) must be satisfied where Δ now denotes the difference between the strip and tree values of the quantities. In addition to there being no net force on any transverse cross-section of a strip there must be no bending moment about two orthogonal axes of such a cross-section. Let the axis of the strip be at $r = r_o$, $\theta = 0$ when it was in the plank and let $x = r\cos\theta - r_o$, $y = r\sin\theta$ then the above conditions may be expressed as:

$$\int_{-h}^{h}\int_{-t}^{t} \sigma_z \, dx\, dy = 0, \quad \int_{-h}^{h}\int_{-t}^{t} \sigma_z x \, dx\, dy = 0 \tag{56}$$
$$\int_{-h}^{h}\int_{-t}^{t} \sigma_z y \, dx\, dy = 0$$

It will now be shown that the trivial stress distribution of no stress throughout is a solution of the stress problem for a thin peripheral strip. Firstly it is clear that the stress equilibrium conditions (equations (1) and equations (56)) and the stress boundary conditions are satisfied if all the stresses are zero. Furthermore, Hooke's law in its inverse form implies that there will be no strain in the strip and thus:

$$\Delta \varepsilon_1 = -\varepsilon_1^{(t)}, \quad \Delta \varepsilon_2 = -\varepsilon_2^{(t)}, \quad \Delta \varepsilon_3 = -\varepsilon_3^{(t)} \tag{57}$$

$$\Delta \gamma_{13} = 0, \quad \Delta \gamma_{12} = 0, \quad \Delta \gamma_{23} = 0$$

Equations (50) may be integrated by using the same procedure as that used for a diametrical plank. This leads to a set of equations identical to equations (53) from which it may be deduced that:

$$\Delta\varepsilon_2 = D_1(r,\theta) = rA_1\sin\theta + rB_1\cos\theta + C_1$$
$$W(r,\theta) = rA_2\sin\theta + rB_2\cos\theta + C_2 \tag{58}$$

$$\Delta\varepsilon_1 = \frac{\partial U}{\partial r}$$
$$\Delta\varepsilon_3 = \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial\theta} \tag{59}$$
$$\Delta\gamma_{13} = \frac{1}{r}\frac{\partial U}{\partial\theta} + r\frac{\partial}{\partial r}(V/r)$$

where U, V and W are related to the change in displacements Δu_1, Δu_2 and Δu_3 by equations (52) and equation (51).

Equations (59) may be integrated by using equations (57) and the expressions for $\varepsilon_1^{(t)}$ and $\varepsilon_3^{(t)}$ (equations (31) and (33)). This leads to the following expressions for U and V:

$$U(r,\theta) = -r\left\{\varepsilon_1^\circ + \frac{\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon}{(p-1)(c_{11}p+c_{13})}\left[\left(\frac{r}{R}\right)^{p-1} - p\right] + \beta\varepsilon\left[\ln\frac{r}{R} - 1\right]\right\} + A_3\sin\theta + B_3\cos\theta \tag{60}$$
$$V(r,\theta) = k\theta r + A_3\cos\theta - B_3\sin\theta + C_3 r$$

where
$$k = \varepsilon_1^\circ - \varepsilon_3^\circ - \beta\varepsilon - \frac{\sigma_3^\circ - \sigma_1^\circ - \delta\varepsilon}{c_{11}p + c_{13}}$$

The constants A_i, B_i, C_i, $(i = 2, 3)$ correspond to rigid displacement without deformation and they may be set equal to zero in the present problem.

Equations (51), (52), (58) and (60) indicate that the displacements within a strip are single-valued functions so that no trouble arises from multi-valuedness. The tangential displacement u_3 in the strip is a linear function of θ but it remains single-valued because a thin peripheral strip will not contain the axis of cylindrical anisotropy; it is therefore not possible to obtain a closed curve within the strip which encloses this axis.

It remains to show that the two expressions for $\Delta\varepsilon_2$ given by equations (57) and (58) are compatible for a thin peripheral strip. Take an origin of coordinates at the axis of the strip with x- and y- axes as indicated by equations (56). The expression for $\varepsilon_2^{(t)}$, equation (32), may then be expanded in a Taylor series about this origin to give:

$$\Delta\varepsilon_2 = -\varepsilon_2^{(t)} = -\varepsilon_2^\circ - \varepsilon\ln\frac{r_0}{R} - \frac{\varepsilon}{r_0}x \tag{61}$$

where terms of order $(\frac{x}{r_0})^2$ and $(\frac{y}{r_0})^2$ or higher have been discarded. This will not introduce serious error because the strip has been assumed to be thin, that is $\frac{h}{R} \ll 1$ and $\frac{t}{R} \ll 1$, and for a peripheral strip $r_0 \approx R$. Since the first of equations (58) may be written in the form:

$$\Delta\varepsilon_2 = A_1 y + B_1(r_0 + x) + C_1$$

the two expressions for $\Delta\varepsilon_2$ are compatible for a sufficiently thin strip.

The above analysis has shown that the trivial stress and strain distribution is a valid solution for a thin peripheral strip cut from a diametrical plank. While the strip is part of the plank its longitudinal strain is given by equation (55) and this strain must therefore be relieved upon cutting. The effects of this strain relief in a strip may be examined by expanding the right-hand side of equation (55) as in the derivation of equation (61). This gives:

$$\varepsilon_2^{(p)}(x, y) = \left(C + \varepsilon_2^{\circ} + \varepsilon \ln \frac{r_o}{R}\right) + \frac{\varepsilon}{r_o} x \qquad (62)$$

Equation (62) implies that the axis of a peripheral strip of in-plank length ℓ will contract upon removal from the plank by an amount:

$$\begin{aligned} \Delta \ell &= \ell \left(C + \varepsilon_2^{\circ} + \varepsilon \ln \frac{r_o}{R} \right) \\ &\approx \ell \left(C + \varepsilon_2^{\circ} - \varepsilon \frac{t}{R} \right) \end{aligned} \qquad (63)$$

since r_o = R-t. Furthermore, if the length of the inner (pith) side of a peripheral strip is ℓ_i after removal and the length of the outer side is ℓ_o, then from equation (62):

$$\left.\begin{aligned} \ell_i &= \ell \left[1 - \varepsilon_2^{(p)}(-t, y) \right] \\ &= \ell \left[1 - C - \varepsilon_2^{\circ} - \varepsilon \ln \frac{r_o}{R} + \frac{\varepsilon}{r_o} t \right] \\ \text{and} \quad \ell_o &= \ell \left[1 - \varepsilon_2^{(p)}(t, y) \right] \\ &= \ell \left[1 - C - \varepsilon_2^{\circ} - \varepsilon \ln \frac{r_o}{R} - \frac{\varepsilon}{r_o} t \right] \end{aligned}\right\} \qquad (64)$$

The strip will therefore bend and from geometrical considerations its radius of curvature will be given by:

$$\begin{aligned} \rho &= \frac{\ell_o + \ell_i}{\ell_i - \ell_o} t \\ &\approx \frac{R}{\varepsilon} \left(1 - \frac{t}{R} \right) \end{aligned} \qquad (65)$$

We are now in a position to compare the theory with experimental data published by JACOBS (1945, 1965). The theory above is applicable to the work of JACOBS since his selection of plank thickness and strip width implies that $\frac{h}{R} = \frac{1}{12}$ and $\frac{t}{R} = \frac{1}{24}$. Firstly, JACOBS (1965) presents some data for Eucalyptus gigantea logs which shows that the radius of curvature of a thin peripheral strip is almost exactly proportional to the radius of the log from which the diametrical plank was cut. This agrees with equation (65). The constant of proportionality is about 480 which suggests that ε in equation (65) is almost equal to 2 x 10^{-3}. Since $\varepsilon \approx 2\ \varepsilon_2^{\circ}$ (see, for example, equations (36)), it follows that the peripheral longitudinal strain in a Eucalyptus gigantea tree is given by the estimate $\varepsilon_2^{\circ} \approx 1 \times 10^{-3}$.

on the variation with log radius of what he calls the "unit contraction" (u.c.) of peripheral strips from a diametrical plank. This quantity is defined in the notation used here by:

$$u.c. = \frac{l_i - l_o}{2t\,l}$$

Using equations (64) this can be expressed as:

$$u.c. = \frac{\varepsilon}{R}\left(1 + \frac{t}{R}\right) \qquad (66)$$

The unit contraction is in fact the radial gradient of the longitudinal strain at the periphery of a tree. By using the method of least squares JACOBS produced the following expression as a curve fitting his unit contraction data:

$$u.c. = \frac{0.00364}{(2R)^{0.925}} \qquad (67)$$

where R is the log radius in inches. The theoretical curve given in equation (66) is very close to that of equation (67), and it gives an equally good fit to the data, provided $\varepsilon \approx 2 \times 10^{-3}$. This leads to the same estimate of ε_2^0 as before.

JACOBS (1945) also measured the change in length of peripheral strips when they were removed from a diametrical plank. He found an average contraction corresponding to a strain of about 8×10^{-4} in his experiments on Eucalyptus gigantea. An examination of equation (63) suggests that C was almost zero in these experiments, that is the longitudinal strain was practically unchanged during the preparation of the diametrical planks. It is of interest to attempt to produce such a value for C by an approximate theoretical approach. Since Figures 2 and 3 suggest that $\sigma_2^{(t)} \approx E_2 \varepsilon_2^{(t)}$ one could assume that the transverse stresses in a diametrical plank also have an insignificant effect on the longitudinal strain and that a similar relationship $\sigma_2^{(p)} \approx E_2 \varepsilon_2^{(p)}$ is valid for the plank. Under this assumption the constant C in equation (55) may be evaluated from the condition that there is no net force on a transverse cross-section of a plank. This leads to the following expression for C:

$$C = \varepsilon\left(1 - \frac{\pi}{4}\frac{h}{R}\right) - \varepsilon_2^0$$

neglecting terms of order $\varepsilon \frac{h^2}{R^2}$ or higher. Since $\varepsilon \approx 2\varepsilon_2^0$ this expression implies that $C \approx \varepsilon_2^0$ which is not consistent with the experimental results. In fact, equation (63) shows that such a value for the constant C would give a fractional contraction of a peripheral strip of $\Delta l / l \approx 2\varepsilon_2^0 \approx 2 \times 10^{-3}$ if the earlier estimates of ε_2^0 are used. This contradiction with experiment is presumably because the assumption $\sigma_2^{(p)} \approx E_2 \varepsilon_2^{(p)}$ is not a valid one.

The results presented here for the longitudinal strain are similar to those of GILLIS (1973). He uses an elastically isotropic model but the in-tree longitudinal strain of his model is in good agreement with that of the model presented in this paper, (see for example, Figure 2). Furthermore the cutting analyses are similar in that both imply that there is a uniform change in the longitudinal strain during the preparation of a diametrical plank from a log. The isotropic model used by GILLIS leads to a uniaxial stress distribution in a diametrical plank and hence GILLIS was able to evaluate $\Delta\epsilon_2$ by the method used earlier to evaluate C. He found that $\Delta\epsilon_2 = \epsilon_2^0$ but he did not notice the contradiction between his theory and experiment that is implied by this value because of an error of a factor of 2 in his expression for the curvature of strips removed from the plank.

CONCLUDING REMARKS

A new model has been presented for the development of growth stresses and strains in trees. It takes into account the strong anisotropic character of wood. The redistribution of the in-tree stresses and strains during various cutting processes has also been investigated. This allows a comparison to be made between the theory and experimental data and in this respect good qualitative agreement has been achieved.

The theoretical stresses and strains have a singularity at the axis of the tree. This is a consequence of allowing the longitudinal and circumferential stresses to be generated in peripheral sheaths of cells even when the radius of the tree is extremely small. In hardwood trees it appears that the peripheral stresses begin to develop after the initial stage of axial growth is completed at a particular height and the tree trunk has begun to grow in diameter only. Furthermore in these early stages the ring structure of the stem does not exist. An initially-unstressed core region of radius r_o which is rectilinearly rather than cylindrically anisotropic may therefore be postulated in the model. An analysis similar to that presented in this paper but applied to the two regions $0 \leq r \leq r_o$ and $r_o \leq r \leq R$ would then show that the growth stresses and strains are independent of r within the core region and have a maximum magnitude which depends on the radius r_o.

In softwood trees axial growth may continue to occur at a particular height for a much longer period than in hardwoods. JACOBS (1965) reports that with <u>Pinus radiata</u> a gradual expansion in longitudinal length between two points may continue for a period of three years, during which period the diameter may increase to three or more inches. This behaviour can produce a compression rather than a tension in the outer layers of a young softwood tree. The present theory is therefore not strictly applicable to softwood trees although the older trees will have a growth stress pattern which is qualitatively similar to the theoretical stress pattern shown in Figure 3.

Even if one takes into account the initially-unstressed core region, the theoretical stress distributions suggest that eventually the stresses near the centre of the tree trunk will become so large that inelastic behaviour of the wood will occur. The occurence of both significant creep and plastic yielding is therefore to be expected in the central portions of the tree. Consequently the theory presented here would be expected to give a good approximation to the real situation in the outer portions of the tree trunk but not in some central portion. Inelasticity will also result in measured strain-releases during experiments which do not correspond to the full amount of strain present in a standing tree.

Finally, it is of interest to ask whether the growth stress phenomenon is incidental or whether it has evolved as a necessary component of the life support of a tree. This author would support the view of KUBLER (1959 a) that the second hypothesis is more likely since the longitudinal tensile stress at the periphery provides the tree trunk with additional strength to resist lateral wind loading. Wood has a much higher tensile strength than compressive strength in the axial direction. Consequently with the outer portion of the tree trunk prestressed in tension, a tree is more capable of withstanding without damage the large compressive strains which will arise on the leeward side during severe winds.

The circumferential compression at the periphery serves the useful purpose of inhibiting the propagation of any cracks produced at the periphery, thereby reducing the risk of the tree drying out. However the peripheral circumferential stress produces transverse tensile stresses at the centre of a tree trunk. Furthermore, the peripheral longitudinal tension generates large compressive stresses at the centre. Either of these effects may cause failure of the wood in the central portion of a tree trunk. The tree therefore protects the sapwood at the expense of the less important heartwood (KUBLER 1959 a).

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FIGURE CAPTIONS

Figure 1 Highly schematic diagram of a segment of a transverse section of a tree trunk, showing the growth region.

Figure 2 Comparison of the in-tree strain distributions of several different theoretical models. The values assumed for the elastic constants are those given in equations (34) and it is assumed that:

$$\varepsilon_2^0 = -\varepsilon_3^0 , \quad \sigma_1^0 = 0 .$$

Figure 3 Comparison of the in-tree stress distributions of several different theoretical models under the same parameter values as in Figure 2.

Figure 4 The theoretical strain distribution in a cross-cut disc under the same parameter values as in Figure 2. Some experimental data from BOYD (1950 a) and KUBLER (1959 a) is also shown.

Figure 5 The theoretical stress distribution in a cross-cut disc under the same parameter values as in Figure 2.

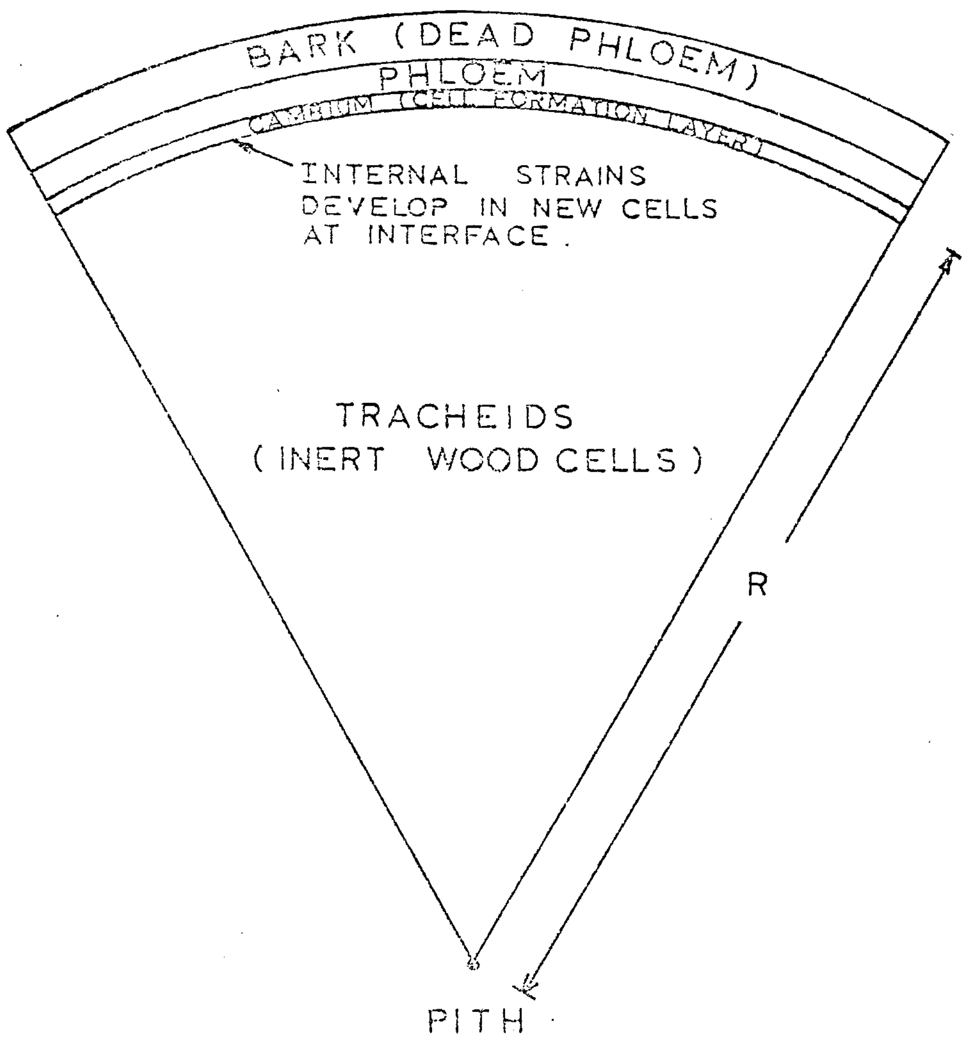
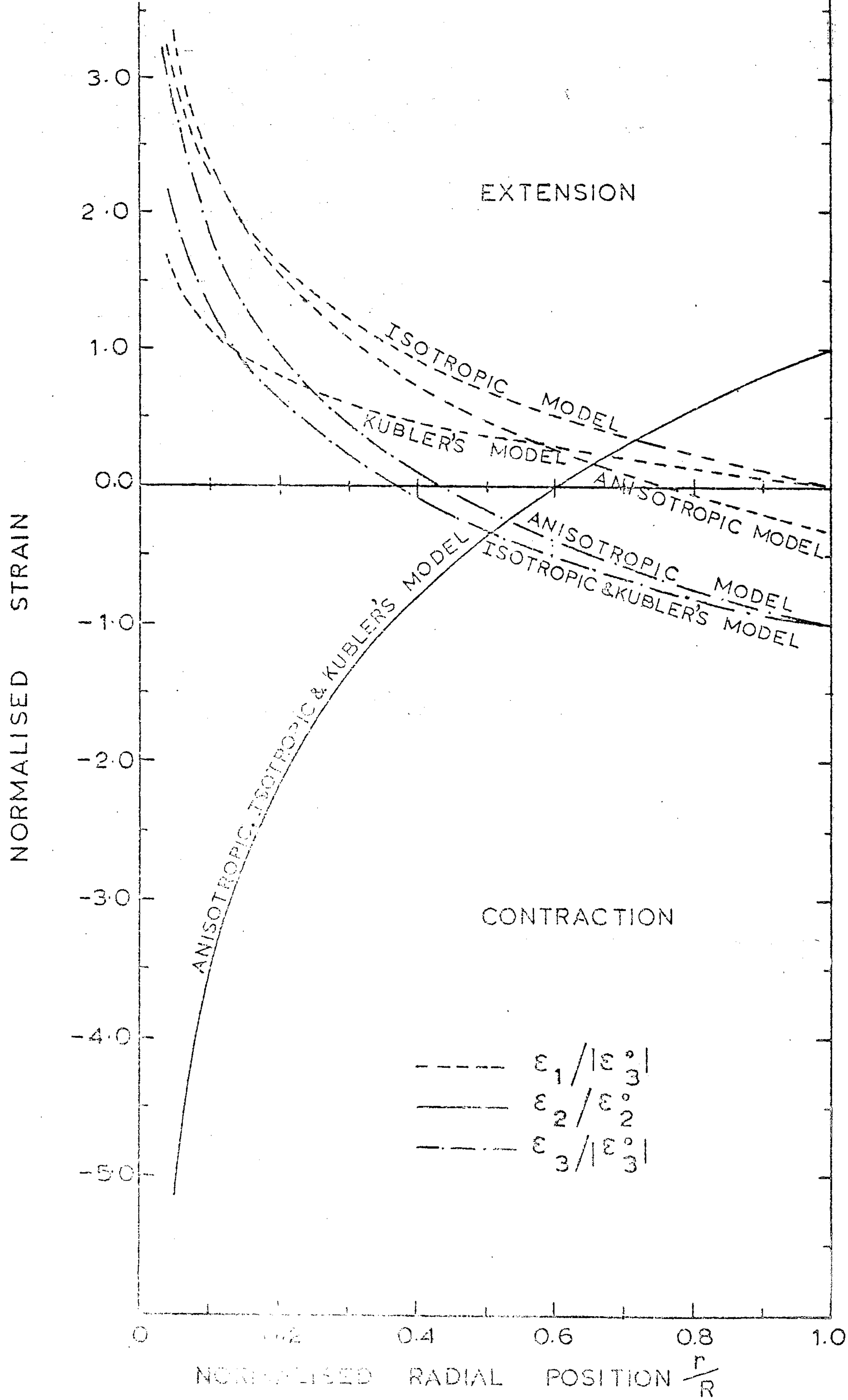

BARK (DEAD PHLOEM)
PHLOEM
CAMBIUM (CELL FORMATION LAYER)
INTERNAL STRAINS DEVELOP IN NEW CELLS AT INTERFACE.
TRACHEIDS
(INERT WOOD CELLS)
R
PITH

FIG. No 1



3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
NORMALISED STRAIN
EXTENSION
ISOTROPIC MODEL
KUBLER'S MODEL
ANISOTROPIC MODEL
ANISOTROPIC MODEL
ISOTROPIC & KUBLER'S MODEL
ANISOTROPIC, ISOTROPIC & KUBLER'S MODEL
CONTRACTION
$\varepsilon_1/|\varepsilon_3^{\circ}|$
$\varepsilon_2/\varepsilon_2^{\circ}$
$\varepsilon_3/|\varepsilon_3^{\circ}|$
0
0.4
0.6
0.8
1.0
RADIAL POSITION r/R

FIG. No 2

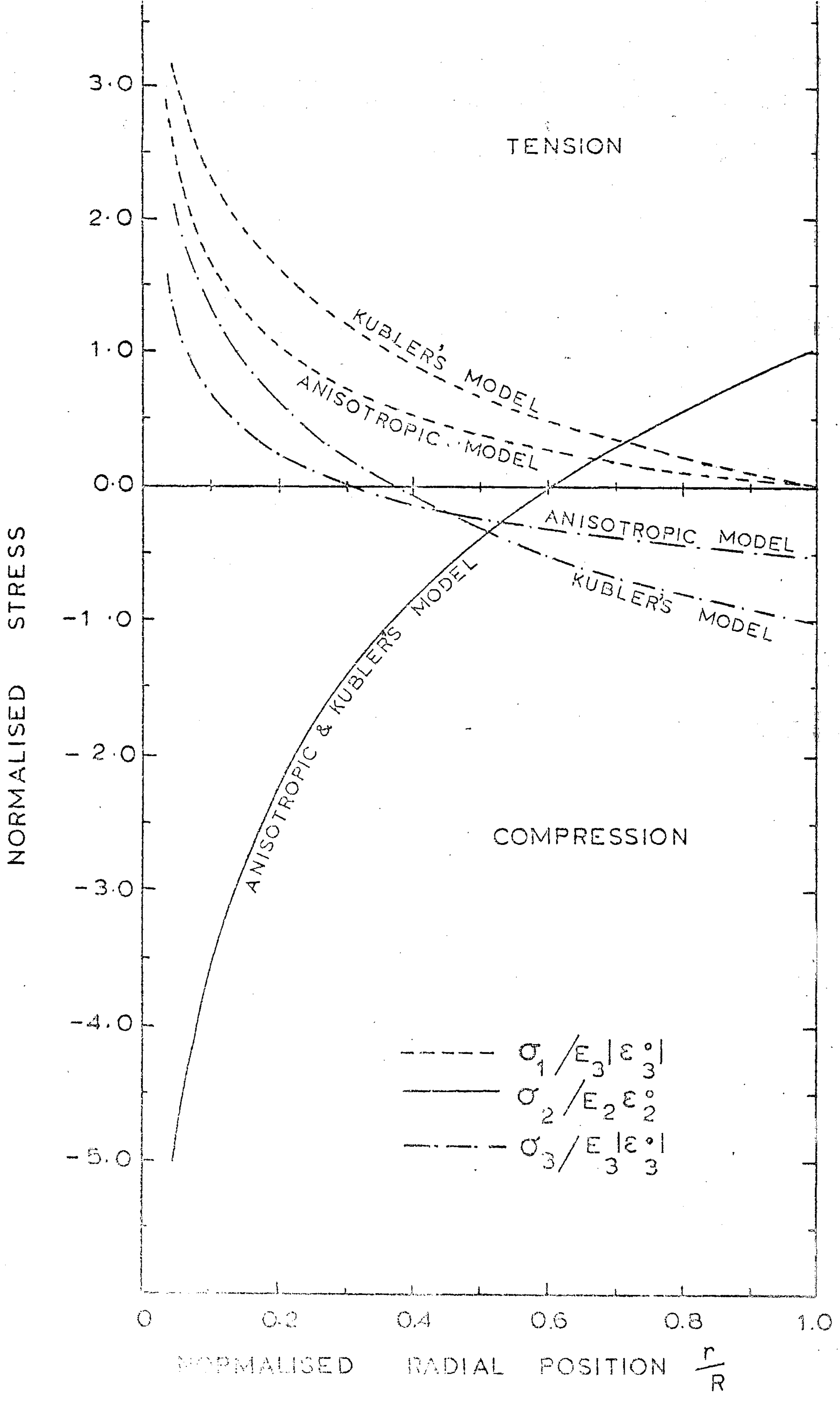

TENSION
COMPRESSION
KUBLER'S MODEL
ANISOTROPIC MODEL
ANISOTROPIC MODEL
KUBLER'S MODEL
ANISOTROPIC & KUBLER'S MODEL
NORMALISED STRESS
NORMALISED RADIAL POSITION $\frac{r}{R}$
3·0
2·0
1·0
0·0
-1·0
-2·0
-3·0
-4·0
-5·0
0
0·2
0·4
0·6
0·8
1·0
$\sigma_1 / E_3 |\varepsilon_3^\circ|$
$\sigma_2 / E_2 \varepsilon_2^\circ$
$\sigma_3 / E_3 |\varepsilon_3^\circ|$

FIG. No 3

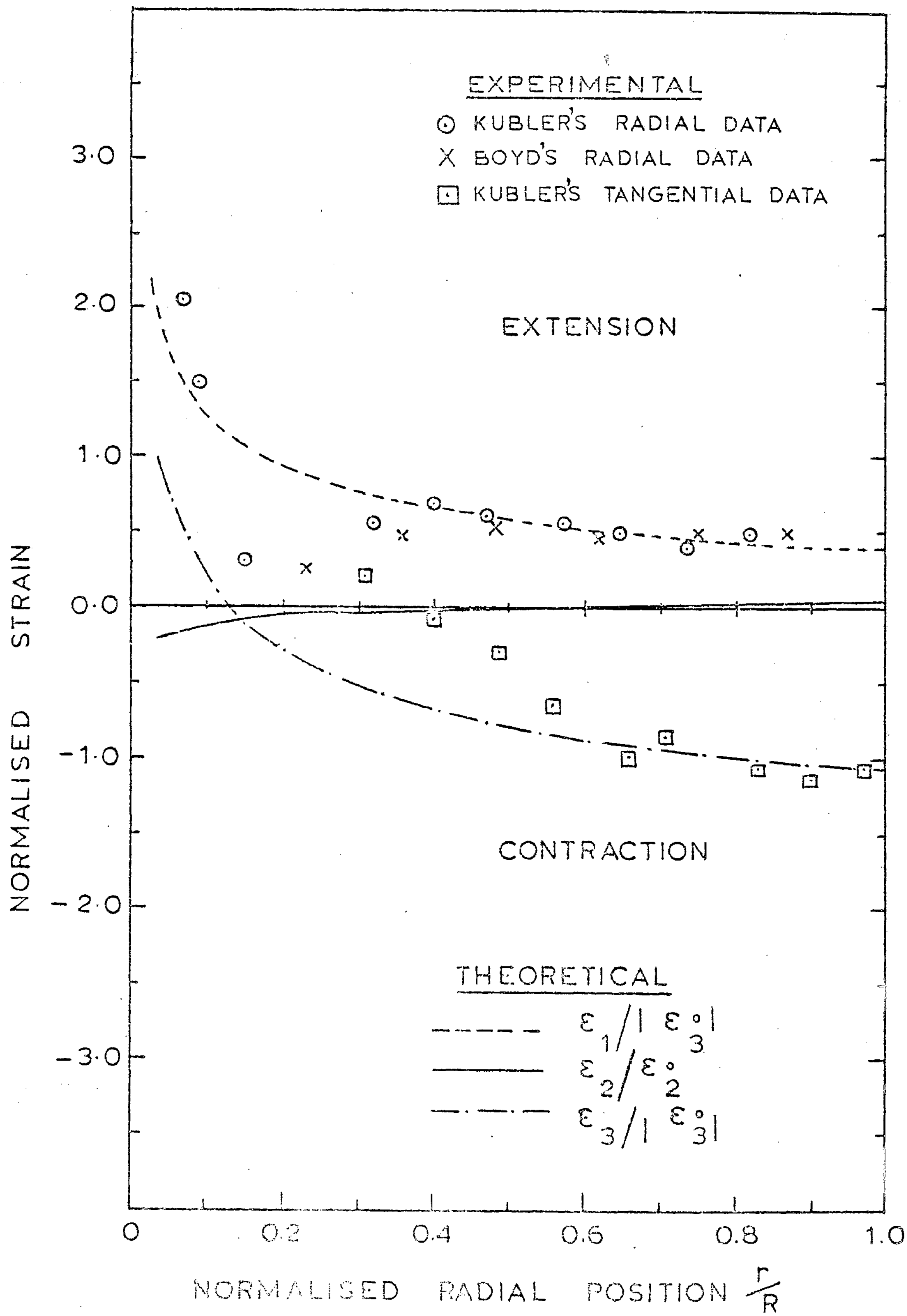
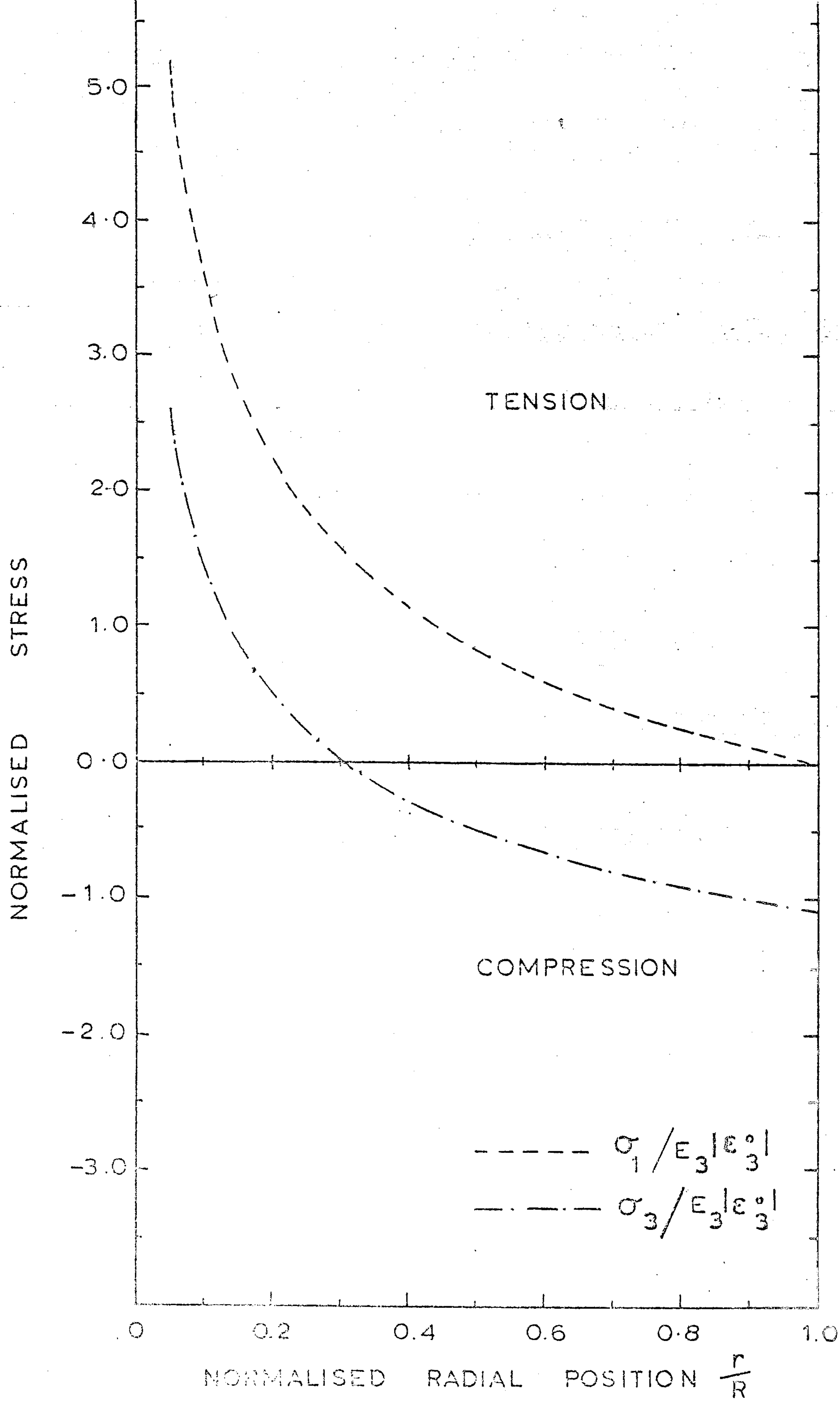

EXPERIMENTAL
KUBLER'S RADIAL DATA
BOYD'S RADIAL DATA
KUBLER'S TANGENTIAL DATA
EXTENSION
CONTRACTION
THEORETICAL
$\varepsilon_1/|\varepsilon_3^\circ|$
$\varepsilon_2/\varepsilon_2^\circ$
$\varepsilon_3/|\varepsilon_3^\circ|$
NORMALISED STRAIN
NORMALISED RADIAL POSITION r/R
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
0
0.2
0.4
0.6
0.8
1.0

FIG. No 4



5·0
4·0
3.0
2·0
1.0
0·0
-1.0
-2.0
-3.0
NORMALISED STRESS
TENSION
COMPRESSION
$\sigma_1 / E_3 |\varepsilon_3^{\circ}|$
$\sigma_3 / E_3 |\varepsilon_3^{\circ}|$
0
0.2
0.4
0.6
0·8
1.0
NORMALISED RADIAL POSITION $\frac{r}{R}$

FIG. No 5